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6 QUATERNIONS APPLIED TO MISSILE SYSTEMS.

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A constructive Euler's Theorem is followed by quaternion representation of missile three-dimensional rotations. The bordered matrix form is emphasized; lead vectors and transmuted equaternions speed computations. The n-th root of a quaternion is the starting point for algebraic recursions that are realized by software on a general purpose digital computer or by dedicated digital computer hardware. Appendices are concerned with error quaternions, four gimbal quaternions, and differential equation representation.

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I. INTRODUCTION

Euler's Theorem is the basis for a four parameter alternative to Euler matrix representations of yaw, roll, and pitch motions of an unbending missile. The quaternion [1-3] has already demonstrated superior features in that:

i) It needs only 35% of the multipliers used by Euler matrix computers.

ii) Analog and digital computations are more accurate than Euler matrix computations.

iii) It is preferred to Euler matrix representation in strap down gyro platforms that replace expensive mechanical gimbals with digital computers.

The bordered matrix form is emphasized exclusively. Transmitted quaternions interchange lead vectors and effect speedier hand computations. These two quaternion implementations are attributed to Ickes [4].

The nth root of a quaternion is an instructive precursor to small angle recursions. These algebraic recursions are regimented by Chebyshev polynomials.

The appendices contain four gimbal quaternions for strap down computers and quaternion representation by differential equations.

II. SOME PRELIMINARIES

Slight familiarity with Euler matrices, trace of a matrix, and quaternions will be presumed. The ordinary matrix-vector notation and some Gibbs vector ($\hat{x}$, $\hat{y}$, $\hat{z}$) algebra will be employed. The transpose of a matrix or vector is denoted by a prime. Unit Gibbs vectors are indispensable and are denoted by a caret; swift interpretations of the vectors ξ and $\hat{\xi}$ are possible. The Euclidian norm $\|\xi\|$ also appears.

<u>Example 1</u>

If ξ is a 3-vector and $\|\xi\| \triangleq \sqrt{\xi'\xi}$, then $\hat{\xi} = \xi/\|\xi\|$. Other objects such as the lead vector and transmitted quaternions will be introduced as needed.

III. ROTATION IN THREE-SPACE

Euler matrices produce rotations in 3-space by successive rotations in three 2-spaces, the xy-plane, the yz-plane, and the zx-plane. Reference vectors are $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively, the first symbols in the plane designations. The plane normals are $\hat{z}$, $\hat{x}$, and $\hat{y}$. The sign of angles is determined by the right hand screw convention. These rotations produce the yaw, roll, and pitch angles, respectively. Euler's theorem returns a 3-space rotation to its basic situation in 2-space.

Theorem 3.1 (Euler). A 3-space rotation is equivalent to a 2-space rotation about a fixed vector. The 2-space rotation has the fixed vector for its normal.

Sketch of Proof:

a) If $E = \{e_{ij}\}$ in a 3×3 Euler matrix such that $EE' = I$, then there exists a fixed 3-vector ξ such that $E\xi = \xi$.

b) The angle of rotation, θ, is given by

$$1 + 2\cos\theta = e_{11} + e_{22} + e_{33} \triangleq \text{tr } E \tag{3.1}$$

c) Alternatively,

$$\cos\frac{\theta}{2} = \frac{1}{2}\sqrt{1 + \text{tr } E} \quad , \tag{3.2}$$

$$\sin\frac{\theta}{2} = \frac{1}{2}\sqrt{\text{tr}(I - E)} \quad . \tag{3.3}$$

The three components of the fixed vector, ξ, and the angle, θ, constitute a four parameter system. Naturally, there exist three possible classes of representations:

i) Four component vectors.

ii) Four parameter bordered matrix.

iii) Four parameter pseudo-vectors.

The first representation will be considered. Afterwards, a return to quaternions in bordered matrix form is in order. The pseudo-vector is sometimes called a hypercomplex number. It is Hamilton's (1843) original formulation and can be found in Bean [3]. It is a matrix decomposition of the quaternion and does not need to be considered in this computation context.

IV. THE CONSTRUCTIVE EULER THEOREM

Theorem 3.1 can be reinforced by two tasks:

i) Determine two vectors in the 2-space whose normal is the fixed vector.

ii) Construct a four-component lead vector that deduces a complete orthogonal basis.

As implied in the statement, a lead vector can deduce other vectors and with them form an orthogonal (orthonormal) bases.

Lemma 4.1. The fixed vector ξ of Theorem 3.1 satisfies $(E - E')\xi = 0$, and its components are

$$\begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} = \begin{pmatrix} e_{23} - e_{32} \\ e_{31} - e_{13} \\ e_{12} - e_{21} \end{pmatrix} .$$

Moreover, there exists a natural vector decomposition

$$\xi = u - v \quad ,$$

where

$$u = \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} ,$$

$$v = \begin{pmatrix} e_{32} \\ e_{13} \\ e_{21} \end{pmatrix} .$$

Lemma 4.2. The vectors $(u \times v)$ and $(u \times v) \times (u - v)$ are in the plane whose normal is the fixed vector ξ.

Studies of four parameter 4-vector representation can commence. The norm preserving mapping

$$(E\xi = \xi) \quad \rightarrow \begin{pmatrix} \|\xi\| \cos\theta \\ \xi \sin\theta \end{pmatrix}$$

is a formal condensation of all that has been stated so far into the 4-vector on the right. The angle θ is implicit in the Euler matrix E through Theorem 3.1. Norm preservation simply means that the 4-sphere has the same norm, $\|\xi\|$, as the 3-sphere.

<u>Lemma 4.3.</u> The vector $\begin{pmatrix} \|\xi\| \cos\theta \\ \xi \sin\theta \end{pmatrix}$ is a lead vector that can deduce the other three members of an orthogonal basis. The orthogonal basis, after using $\xi = u - v$, is

$$\begin{pmatrix} \|u-v\| \cos\theta \\ (u-v)\sin\theta \end{pmatrix}, \begin{pmatrix} -\|u-v\| \sin\theta \\ (u-v)\cos\theta \end{pmatrix}, \begin{pmatrix} 0 \\ u \times v \end{pmatrix}, \begin{pmatrix} 0 \\ (u-v)\times(u \times v) \end{pmatrix} .$$

If $c \triangleq u \times v$, $d \triangleq (u \times v) \times (u - v)$, then the orthonormal basis is

$$\begin{pmatrix} \cos\theta \\ \hat{\xi} \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \hat{\xi} \cos\theta \end{pmatrix}, \begin{pmatrix} 0 \\ \hat{c} \end{pmatrix}, \begin{pmatrix} 0 \\ \hat{d} \end{pmatrix} .$$

The second vector is obtained by replacing θ in the first vector with $\theta + (\pi/2)$. No more angles are available, and the top components of the third and fourth vector are zero. The scalar triple product from Gibbs' vector algebra ensures orthogonality of the vector parts of the third and fourth basis vectors.

Lemmas 3.1, 4.1, 4.2, and 4.3 comprise the Constructive Euler Theorem whose corollary is an orthogonal basis of 4-vectors. The Constructive Euler Theorem generates the need for 4-vectors, but computations are slow. Swift computation is effected by quaternions largely because cross products are present in quaternion products.

V. SIMPLE QUATERNION COMPUTATIONS

The quaternion can be introduced through the nonstandard Cayley-Klein form

$$\begin{pmatrix} a + jb & -c + jd \\ c + jd & a - jb \end{pmatrix}, \quad j^2 = -1 \quad .$$

A nonunique matrix representation of complex numbers leads to the 4×4 matrix

$$\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} = Q(q), \quad q = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \qquad (5.1)$$

The notation $Q(q)$ means the "quaternion array" Q whose first column vector is q. The standard Cayley-Klein form places the lead vector in the the first row. After becoming familiar with fundamentals, a bordered 4×4 matrix will be used exclusively; this usage coincides with Ickes [4].

An intermediate result is needed.

<u>Lemma 5.1</u>. The column vectors of $Q(q)$ form an orthogonal basis; the column vectors of $Q(\hat{q})$ form an orthonormal basis. Moreover, q is a lead vector.

<u>Proof:</u>

a) $$Q(q)\, Q'(q) = (q'q)\, I \qquad (5.2)$$

is obtained by direct computation.

b) $$Q(\hat{q})\, Q'(\hat{q}) = I. \qquad (5.3)$$

This lemma immediately replaces many geometrical calculations required by the orthogonal basis of the Constructive Euler Theorem with a format of four signed and unsigned rearrangements of the lead vector q that are orthogonal.

If indices are employed for the lead vector, the nonstandard vector component notation $(p_0, p_1, p_2, p_3)'$ will be employed. Oftentimes interest focusses on the "vector part" $(p_1, p_2, p_3)'$ and the lead vector is

shortened to $(p_0, p)'$; this partitioned vector reduces symbol clutter. The "real part" of the vector is the scalar p_0.

Skew quaternions with zero real part are interesting in their own right. Direct computation gives

$$Q\begin{pmatrix}0\\p\end{pmatrix}\begin{pmatrix}0\\q\end{pmatrix} = \begin{pmatrix}q \cdot p\\q \times p\end{pmatrix} = \begin{pmatrix}p'q\\Pq\end{pmatrix},$$

$$P = \begin{bmatrix}0 & p_3 & -p_2\\-p_3 & 0 & p_1\\p_2 & -p_1 & 0\end{bmatrix} \tag{5.4}$$

depending on whether Gibbs' vector or matrix — vector notations are employed. A computation such as the commutator identity

$$\frac{1}{2}\left[Q\begin{pmatrix}p_0\\p\end{pmatrix}Q\begin{pmatrix}q_0\\q\end{pmatrix} - Q\begin{pmatrix}q_0\\q\end{pmatrix}Q\begin{pmatrix}p_0\\p\end{pmatrix}\right] = Q\begin{pmatrix}0\\Pq\end{pmatrix} \tag{5.5}$$

requires more tools for effective computation. Two digressions are needed to consider lead vector selection rules and transmuted quaternions.

Finally, the bordered matrix form

$$Q\begin{pmatrix}p_0\\p\end{pmatrix} = \begin{pmatrix}p_0 & -p'\\p & p_0 I + P\end{pmatrix} \tag{5.6}$$

exhibits the 3×3 matrix, $(p_0\, I + P)$, the "kernel" matrix of the quaternion. If $p_0 = 0$, then the skew matrix P is the kernel of $Q\begin{pmatrix}0\\p\end{pmatrix}$. The symmetric part of $Q\begin{pmatrix}p_0\\p\end{pmatrix}$ is $Q\begin{pmatrix}p_0\\\vec{0}\end{pmatrix} = p_0\, I$. It is important to realize that $\begin{pmatrix}p_0\\p\end{pmatrix}$ automatically constructs the full quaternion.

Interaction of real part, vector part, and kernel should be apparent.

VI. LEAD VECTOR SELECTION RULES

Lead vector selection rules transform matrix multiplications to simpler matrix-vector multiplications and thereby reduce calculation effort.

Let Dirac brackets $\langle \cdot \rangle$ denote the selection of the first column vector from a given matrix; i.e.,

$$\langle Q(p) \rangle = p \quad . \tag{6.1}$$

The following rules can be deduced from matrix multiplication rules:

i) Distributivity over sums:

$$\langle Q(p) + Q(q) \rangle = \langle Q(p) \rangle + \langle Q(q) \rangle = p + q \quad , \tag{6.2}$$

ii) Distributivity over products:

$$\langle Q(p)\ Q(q) \rangle = \langle Q(p)\ \langle Q(q) \rangle \rangle \tag{6.3}$$

iii) Absorption rule:

$$\langle Q(p)\ q \rangle = Q(p)\ q \tag{6.4}$$

iv) Idempotent rule:

$$\langle \langle Q(p) \rangle \rangle = \langle p \rangle = p \tag{6.5}$$

The idempotent rule is a slight extension of the absorption rule for the case where Q is replaced by the identity matrix. In the distributivity rule, the flow of Dirac brackets is from outside to inside and from left to right.

The Q operation in

$$Q(Q(p)\ q) = Q(p)\ Q(q) \tag{6.6}$$

is the inverse of the absorption rule.

Many of these rules are general in that they can be applied to matrices and vectors which are not quaternions and lead vectors.

VII. TRANSMUTED QUATERNIONS

The second requirement for converting matrix multiplications is the interchange of lead vectors on the left-hand side of our similarity equation

$$Q(q)\ p = Q(\bar{p})\ q \tag{7.1}$$

to the preferred lexicographical order $\bar{p}$ q on the right-hand side.

If $p' \triangleq (a, b, c, d)$ and $q' \triangleq (\alpha, \beta, \gamma, \delta)$,

then form

$$Q(p)\ q = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha a - \beta b - \gamma c - \delta d \\ \beta a + \alpha b - \delta c + \gamma d \\ \gamma a + \delta b + \alpha c - \beta d \\ \delta a - \gamma b + \beta c + \alpha d \end{pmatrix} . \tag{7.2}$$

Rewrite the rightmost vector so that Greek letters are in a matrix and Latin letters are in the column vector to obtain

$$Q(p)\ q = \begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \triangleq Q_7(q)\ p \quad . \tag{7.3}$$

If one suppresses the distinction between Greek and Latin letters, the array Q_7 is obtained from Q by transposing only the kernel matrix. $Q_7(q)$ is the transmuted quaternion and one can verify that q is a lead vector.

The matrix-vector result can be summarized.

<u>Lemma 7.1.</u>

$$Q(q)\ p = Q_7(p)\ q$$

$$\left[Q_7(p)\right] \left[Q_7(p)\right]' = (p'p)\ I \tag{7.4}$$

$$\text{tr}\ Q_7(p) = \text{tr}\ Q(p)$$

If the problem is interchanging p and q in $p'\,Q(q)$, one then has the dual problem that leads to the second transmuted quaternion. Form

$$p'\,Q(q) = (a \quad b \quad c \quad d)\begin{pmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & \delta & -\gamma \\ \gamma & -\delta & \alpha & \beta \\ \delta & \gamma & -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha a + \beta b + \gamma c + \delta d \\ \alpha b - \beta a + \gamma d - \delta c \\ \alpha c - \beta d - \gamma a + \delta b \\ \alpha d + \beta c - \gamma b - \delta a \end{pmatrix} . \tag{7.5}$$

Rewrite the rightmost vector so that Greek letters are in the row vector and Latin letters are in the matrix to obtain

$$p'\,Q(q) = (\alpha \quad \beta \quad \gamma \quad \delta)\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix} \triangleq q'\,Q_1(p) \tag{7.6}$$

If one suppresses the Greek and Latin letters, the array Q_1 is obtained from Q by multiplying the second, third, and fourth column vectors by -1. Moreover, p is the lead vector of $Q_1(p)$, which is the second transmuted quaternion.

This matrix-vector result can be summarized.

Lemma 7.2.

$$q'\,Q(p) = p'\,Q_1(q)$$

$$\left[Q_1(p)\right]\left[Q_1(p)\right]' = (p'p)\,I \quad . \tag{7.7}$$

$$\mathrm{tr}\,Q_1(p) = -0.5\ \mathrm{tr}\,Q(p) \quad .$$

Ickes [4] employed $Q^{\neq}$ in place of our notation Q_7 and called $Q^{\neq}$ the transmuted quaternion. This notion is generalized in Appendix A.

VIII. FROM QUATERNIONS TO EULER MATRICES

This section reinforces previous methods to achieve firmer foundations. The first task is to present methods from behind the scenes of Lemmas 3.1 and 4.1.

Lemma 8.1. Given any 3-vector, k, it's annihilator K, a 3- × 3-skew matrix, can be constructed as follows

$$Kk = \begin{pmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} k_3k_2 - k_2k_3 \\ k_3k_1 + k_1k_3 \\ k_2k_1 - k_1k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \qquad (8.1)$$

Moreover, if g is another 3-vector, then

$$Kg = -Gk = g \times k,$$

$$K' = K,$$

$$G' = -G,$$

$$\mathrm{tr}\, K^2 = -2k'k,$$

$$K^3 + (k'k)\, K = 0.$$

Lemma 8.2. If $h = Kg$ and $K' = -K$, then $H = KG - GK = gk' - kg'$. Proof is by direct calculation.

A frequent problem is that K is known but ξ is unknown and subject to $K\xi = 0$. The solution is $\xi = k$ up to an arbitrary non-zero scalar factor.

Note that the trace operation, trA, as the sum of the diagonal elements of the matrix A.

The second task is to derive a more specific form of the transposed lead vector $(\cos\theta, \hat{\xi}' \sin\theta)$.

The third task is to derive the Euler matrix from a quaternion. The second and third problems are solved together.

Theorem 8.3. If $\hat{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\hat{\rho} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$

and $\begin{pmatrix} p_0 \\ p \end{pmatrix}$ is a unit lead vector, then the similarity transformation

$$Q'\begin{pmatrix} p_0 \\ p \end{pmatrix} Q\begin{pmatrix} 0 \\ \hat{r} \end{pmatrix} Q\begin{pmatrix} p_0 \\ p \end{pmatrix} = Q\begin{pmatrix} 0 \\ \hat{\rho} \end{pmatrix} \tag{8.2}$$

$$p_0^{\;2} + p'p = 1$$

induces the Euler transformation

$$\hat{\rho} = \left[pp' + (p_0 I - P)^2\right] \hat{r} \triangleq E\hat{r} \tag{8.3}$$

and

$$p_0 = \pm\left|\cos\left(\frac{\theta}{2}\right)\right|$$

p = the fixed vector of E.

<u>Proof:</u>

i) $Q'\begin{pmatrix} p_0 \\ p \end{pmatrix} Q_7\begin{pmatrix} p_0 \\ p \end{pmatrix}\begin{pmatrix} 0 \\ \hat{r} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\rho} \end{pmatrix}$

ii) $\begin{bmatrix} p_0 & p' \\ -p & p_0 I - P \end{bmatrix} \begin{bmatrix} p_0 & -p' \\ p & p_0 I - P \end{bmatrix} \begin{bmatrix} 0 \\ \hat{r} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\rho} \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 0' \\ \vec{0} & pp' + (p_0 I - P)^2 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{r} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{\rho} \end{bmatrix}$

iv) $\mathrm{tr}\left(pp' + (p_0 I - P)^2\right) = 3\,p_0^{\;2} - p'p = 1 + 2\cos\theta$

implies that

$$\left[p_0^{\;2} = \left(\cos\frac{\theta}{2}\right)^2\right] \text{ or } \left[p_0 = \pm\left|\cos\left(\frac{\theta}{2}\right)\right|\right]$$

v) $\left[Q'(p)\,Q_7(p)\right]'\left[Q'(p)\,Q_7(p)\right] = I$ implies that $E'E = I$

vi) ($E\xi = \xi$ and $EE' = I$) imply that $(E - E')\xi = 0$

vii) $(E - E')\xi = -4\,p_0 P\xi = 0$, $p_0 \neq 0$, implies $\xi = p$ up to a multiplicative scalar factor;

viii) $\begin{pmatrix} p_0 \\ p \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \hat{p}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$.

Step viii displays an amended, principal part form for a lead vector that originally contained the full angle. Note that the ± sign on p_0 in Step iv has been arbitrarily made positive in Step viii. This sign decision must be made external to the quaternion system.

Some straightforward calculations can be executed. The first calculation is the explicit form of the Euler matrix, E, in terms of the lead vector (p_0, p') components; namely

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} 1 - 2p_2^2 - 2p_3^2 & 2p_1p_2 - 2p_0p_3 & 2p_0p_2 + 2p_1p_3 \\ 2p_0p_3 + 2p_1p_2 & 1 - 2p_1^2 - 2p_3^2 & 2p_2p_3 - 2p_0p_1 \\ 2p_1p_3 - 2p_0p_2 & 2p_0p_1 + 2p_2p_3 & 1 - 2p_1^2 - 2p_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} . \tag{8.4}$$

The skew matrix is

$$\begin{bmatrix} 0 & -4p_0p_3 & 4p_0p_2 \\ 4p_0p_3 & 0 & -4p_0p_1 \\ -4p_0p_2 & 4p_0p_1 & 0 \end{bmatrix} = -4p_0 \begin{bmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{bmatrix} . \tag{8.5}$$

The second and last calculation specifies a product of quaternions

$$Q\begin{pmatrix} p_0 \\ p \end{pmatrix} = Q\begin{pmatrix} u_0 \\ u \end{pmatrix} Q\begin{pmatrix} v_0 \\ v \end{pmatrix} Q\begin{pmatrix} w_0 \\ w \end{pmatrix} , \tag{8.6}$$

where u, v, and w are mutually orthogonal vector parts to greatly reduce calculations. Employing $\hat{x}$, $\hat{y}$, $\hat{z}$ for the first, second, and third column vectors of the identity matrix, one obtains the specific form

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \hat{p}\sin\frac{\theta}{2} \end{pmatrix} = Q\begin{pmatrix} \cos\frac{\mu_1}{2} \\ \hat{z}\sin\frac{\mu_1}{2} \end{pmatrix} Q\begin{pmatrix} \cos\frac{\mu_2}{2} \\ \hat{x}\sin\frac{\mu_2}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\mu_3}{2} \\ \hat{y}\sin\frac{\mu_3}{2} \end{pmatrix} . \tag{8.7}$$

The strange unit vector and subscript association derives from our canonical Euler transformation sequence; i.e., rotation first in the xy-plane whose normal is $\hat{z}$, rotation in the yz-plane whose normal is $\hat{x}$, and rotation in the zx-plane whose normal is the $\hat{y}$. Moreover, the leftmost quaternion performs the first operation in the similarity transformation. Direct calculation leads to

$$p_0 = \cos\left(\frac{\theta}{2}\right) = \left(\cos\frac{\mu_1}{2}\right)\left(\cos\frac{\mu_2}{2}\right)\left(\cos\frac{\mu_3}{2}\right) + \left(\sin\frac{\mu_1}{2}\right)\left(\sin\frac{\mu_2}{2}\right)\left(\sin\frac{\mu_3}{2}\right) \tag{8.8}$$

$$\hat{p} = \left(\csc\frac{\theta}{2}\right)\begin{pmatrix} \cos\frac{\mu_1}{2} & \sin\frac{\mu_2}{2} & \cos\frac{\mu_3}{2} & + & \sin\frac{\mu_1}{2} & \cos\frac{\mu_2}{2} & \sin\frac{\mu_3}{2} \\ \cos\frac{\mu_1}{2} & \cos\frac{\mu_2}{2} & \sin\frac{\mu_3}{2} & - & \sin\frac{\mu_1}{2} & \sin\frac{\mu_2}{2} & \cos\frac{\mu_3}{2} \\ \sin\frac{\mu_1}{2} & \cos\frac{\mu_2}{2} & \cos\frac{\mu_3}{2} & - & \cos\frac{\mu_1}{2} & \sin\frac{\mu_2}{2} & \sin\frac{\mu_3}{2} \end{pmatrix} . \tag{8.9}$$

It should be emphasized here that the quaternion angles (μ_1, μ_2, μ_3) are the negative Euler angles; this anomaly arises because of our choice of the first column vector as the lead vector.

IX. SMALL ANGLE APPROXIMATIONS

The small angle approximations

$$\left.\begin{array}{l}\sin\mu \doteq \mu \\ \cos\mu \doteq 1\end{array}\right\} \text{ for } |\mu| \lesssim 0.1 \text{ rad}$$

is a mechanism for calculating the Euler matrix, E, from the following identity. If

$$K = \begin{pmatrix} 0 & \mu_3 & -\mu_2 \\ -\mu_3 & 0 & \mu_1 \\ \mu_2 & -\mu_1 & 0 \end{pmatrix} \quad ,$$

then

$$\exp\begin{bmatrix} 0 & 0' \\ \vec{0} & K \end{bmatrix} = \begin{bmatrix} 1 & 0' \\ \vec{0} & \exp K \end{bmatrix} = \begin{bmatrix} 1 & 0' \\ \vec{0} & E \end{bmatrix} \quad . \tag{9.1}$$

The exponential matrix is also the culmination of the differential equation formulation of rotations (Appendix C). Although differential equations can be solved recursively, one would rather emphasize direct algebraic methods.

The next task is to compare small angle second-order effects on Euler matrices and quaternions. Consider the quaternion similarity transformation again. Quaternion calculations specifically lead to the Euler matrix

$$pp' + (p_0 I - P)^2 = I - (\sin\theta)\hat{P} + \left(\sin\frac{\theta}{2}\right)^2 (\hat{p}\hat{p}' + P^2 - I) \quad . \tag{9.2}$$

Some simplified calculations are in order. If

$$\hat{p}' = (1, 0, 0)$$

and

$$\bar{p}' \triangleq (0, 1, 1) \quad ,$$

then

$$\hat{p}\hat{p}' + \hat{P}^2 - I = -2 \operatorname{diag}(\bar{p}) \quad .$$

If the previous identity and the small angle approximation are invoked, one obtains

$$pp' + (p_0 I - P)^2 = I - \theta\hat{P} - \frac{\theta^2}{2} \operatorname{diag}(\overline{p}) \quad . \tag{9.3}$$

However, if

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \hat{p}\sin\frac{\theta}{2} \end{pmatrix} \doteq \begin{pmatrix} 1 \\ p\,\frac{\theta}{2} \end{pmatrix} ,$$

then

$$pp' + (p_0 I - P)^2 = I - \theta\hat{P} - \left(\frac{\theta}{2}\right)^2 T \quad , \tag{9.4}$$

$$T = \operatorname{diag}(1, -1, -1),$$

and matrix norm comparisons of the second-order matrices can be made; these are

$$\left\| \frac{\theta^2}{2} \operatorname{diag}(\overline{p}) \right\| = \frac{\sqrt{2}}{2}\,\theta^2 = 0.707\theta^2 \tag{9.5}$$

$$\left\| \frac{\theta^2}{4} T \right\| = \frac{\sqrt{3}}{4}\,\theta^2 = 0.433\theta^2 \quad . \tag{9.6}$$

One can conclude that the quaternion small angle approximation should be slightly better than the Euler matrix small angle approximation if the goal is to obtain the smallest second-order terms. Such a goal is implicit in the exponential matrix, Equation (9.1).

X. NTH ROOT OF A QUATERNION

Before venturing into small angle-induced recursions, it is instructive to find the nth root of a quaternion. This provides an overview of the digital recursion scene irrespective of the large computation effort implied.

Three mathematical tools must be used. These are as follows:

a) The Cayley-Hamilton Theorem.

b) The minimal polynomial equation of a quaternion.

c) The modified Chebyshev polynomials $S_n(x)$ and $C_n(x)$.

Lemma 10.1. Cayley-Hamilton – The polynomial equation of $Q\begin{pmatrix} p_0 \\ p \end{pmatrix}$ is

$$\left[Q^2 - 2p_0\, Q + \left(p_0^2 + p'p\right) I\right]^2 = 0 \tag{10.1}$$

and the minimal polynomial equation for a unit quaternion is

$$Q^2 = 2p_0\, Q - I \quad . \tag{10.2}$$

This minimal equation directly links the modified Chebyshev polynomial $S_n(x)$ with this subject through powers of a quaternion in the identity

$$Q^n\begin{pmatrix} p_0 \\ p \end{pmatrix} = S_{n-1}(2p_0)\; Q\begin{pmatrix} p_0 \\ p \end{pmatrix} - S_{n-2}(2p_0)\; Q\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} \tag{10.3}$$

$$n = 2, 3, 4, 5, \ldots \quad .$$

The right-hand side of the equation can be written as

$$Q^n\begin{pmatrix} p_0 \\ p \end{pmatrix} = Q\begin{pmatrix} p_0 S_{n-1}(2p_0) - S_{n-2}(2p_0) \\ p S_{n-1}(2p_0) \end{pmatrix} \tag{10.4}$$

or

$$Q^n\begin{pmatrix} p_0 \\ p \end{pmatrix} = Q\begin{pmatrix} 0.5\ C_n(2p_0) \\ p\ S_{n-1}(2p_0) \end{pmatrix} \quad . \tag{10.5}$$

The last equation exhibits the remaining Chebyshev polynomial $C_n(x)$. Lanczos [5] presents a very readable introduction to Chebyshev polynomials.

Chebyshev polynomial identities,

$$C_n(2\cos\theta) = 2\cos n\theta \tag{10.6}$$

and

$$S_{n-1}(2\cos\theta) = \frac{(\sin n\theta)}{(\sin\theta)} \quad , \tag{10.7}$$

strongly suggest the form of the nth root of a quaternion.

<u>Lemma 10.2.</u> If n is an integer and $n \geq 2$, then the nth root of $Q\begin{pmatrix} \cos\theta \\ \hat{p}\sin\theta \end{pmatrix}$ is $Q\begin{pmatrix} \cos(\theta/n) \\ p\ \sin(\theta/n) \end{pmatrix}$.

<u>Proof</u>:

$$\left[Q\begin{pmatrix} \cos\left(\frac{\theta}{n}\right) \\ \hat{p}\ \sin\left(\frac{\theta}{n}\right) \end{pmatrix}\right]^n = Q\begin{pmatrix} 0.5\ C_n\left(2\cos\frac{\theta}{n}\right) \\ \hat{p}\left(\sin\frac{\theta}{n}\right) S_{n-1}\left(2\cos\frac{\theta}{n}\right) \end{pmatrix} = Q\begin{pmatrix} \cos\theta \\ \hat{p}\ \sin\theta \end{pmatrix} \quad . \tag{10.8}$$

In the similarity transformation context, the full angle θ must be replaced with $\theta/2$.

Computation complexity increases when $\cos(\theta/n)$ and $\sin(\theta/n)$ must be computed from $\cos\theta$ and $\sin\theta$. Trigonometric computations first yield

$$\cos\left(\frac{\theta}{2}\right) = 0.5\sqrt{1 + \cos\theta}$$

and

$$\cos\left(\frac{\theta}{4}\right) = 0.5\sqrt{1 + \cos\frac{\theta}{2}}$$

and, in general, one obtains the recursion

$$\cos\frac{\theta}{2^{m+1}} = 0.5\sqrt{1 + \cos\left(\frac{\theta}{2^m}\right)}$$

$$m = 0, 1, 2, 3, \ldots \quad . \tag{10.9}$$

Finally, trigonometric computations yield

$$\sin\frac{\theta}{2^{m+1}} = 0.5\sqrt{2 + \sin^2\left(\frac{\theta}{2^m}\right)}$$

$$m = 0, 1, 2, 3, \ldots \quad . \tag{10.10}$$

Unfortunately, iterated square roots in both computations require multiple precision computations.

The nth root of a quaternion demonstrates that the unit quaternion property must be maintained in a recursion. Limitations inherent in $\cos(\theta/n)$ and $\sin(\theta/n)$ calculations and in the fact that n must be a power of two can be avoided by the Padé approximant to be presented in the next section.

With increasing n, 1/n tends to zero. If continued further to n = -1, then the inverse problem reoccurs as presented in Appendix B.

XI. SMALL ANGLE RECURSIONS

The small angle approximations are restarted with the norm preserving Padé approximant

$$e^{j(\theta/2n)} \doteq \frac{4n + j\theta}{4n - j\theta} \quad , \quad \left|\frac{\theta}{2n}\right| \leq 0.1 \text{ rad} \quad , \quad j^2 = -1 \quad , \qquad (11.1)$$

which implies that

$$\cos\frac{\theta}{2n} \doteq \frac{16n^2 - \theta^2}{16n^2 + \theta^2} \qquad (11.2)$$

and

$$\sin\frac{\theta}{2n} \doteq \frac{8n\theta}{16n^2 + \theta^2} \quad . \qquad (11.3)$$

If $\theta = 1$ rad $\doteq 53$ deg and $n = 5$, then $|\theta/2n|$ is at the equality boundary. In this case,

$$\cos\left(\frac{1}{n}\right) \doteq \frac{399}{401} \quad ,$$

$$\sin\left(\frac{1}{n}\right) \doteq \frac{40}{401} \quad ,$$

and the computer binary words must contain at least 9 bits.

The nth root of $Q\begin{pmatrix} \cos(\theta/2) \\ \hat{p}\ \sin(\theta/2) \end{pmatrix}$ is approximated by

$$\frac{1}{16n^2 + \theta^2}\, Q\begin{pmatrix} 16n^2 - \theta^2 \\ 8n\theta\ \hat{p} \end{pmatrix} \triangleq A \quad .$$

The finite matrix-vector iteration format is

$$q_o = \langle A \rangle = \frac{1}{16n^2 + \theta^2} \begin{pmatrix} 16n^2 - \theta^2 \\ 8n\theta\ \hat{p} \end{pmatrix}$$

$$q_m = A\ q_{m-1}$$

$$m = 1, 2, 3, \ldots, (n - 1) \quad . \tag{11.4}$$

Normalization is enforced by the iteration scheme

$$q_1 = \langle A \rangle$$

$$q_m = A\ q_{m-1}$$

$$q_{m+1} = \frac{q_m}{\sqrt{q_m' \, q_m}}$$

$$m = 2, 4, 6, \ldots, (2n - 2)$$

$$q_i = \text{4-vectors} \quad . \tag{11.5}$$

Figure 1 presents the computer flow chart for this recursion.

The end result

$$Q(q_{n-1}) = A^n = Q\begin{pmatrix} p_0 \\ p \end{pmatrix}$$

is given by

$$p_0 = \frac{1}{2} C_n\left(\frac{2(16n^2 - \theta^2)}{(16n^2 + \theta^2)}\right) \triangleq \gamma_n(\theta) \quad , \tag{11.6}$$

$$p = \hat{p}\ \frac{8n\theta}{16n^2 + \theta^2}\ S_{n-1}\left(\frac{32n\theta}{16n^2 + \theta^2}\right) \triangleq \hat{p}\ \sigma_n(\theta) \quad . \tag{11.7}$$

Entering these values into the Euler matrix and using the identity

$$\gamma_n^2 + \sigma_n^2 = 0.25\ C_n(2x) + (1 - x^2)\ S_{n-1}^2(2x) = 1 \tag{11.8}$$

yields a slight simplification in

$$pp' + (p_0 I - P)^2 = I - 2\gamma_n\sigma_n\ \hat{P} + \sigma_n^2(\hat{p}\hat{p}' + \hat{P}^2 - I) \quad . \tag{11.9}$$

A slight hazard occurs when $C_n(2p_0) = 0$; this results in much computation to produce a simple result, $Q\begin{pmatrix}0\\ \hat{p}\end{pmatrix}$.

These end results suggest that dedicated digital computer iterations may be emulated by the iterative processes of $C_n(2p_0)$ and $\sqrt{1 - p_o^2}\ S_n(p_0)$; thereby, matrix-vector iterations and normalizations are minimized. The sole matrix-vector calculation appears in the quaternion to Euler matrix conversion near the end.

The minimum multiplication recursion for the modified Chebyshev polynomials [5] is

$$S_0(x) = 1$$

$$S_1(x) = x$$

$$S_n(x) = x\ S_{n-1}(x) - S_{n-2}(x) \quad , \tag{11.10}$$

followed by additions in

$$C_0(x) = 2$$

$$C_1(x) = x$$

$$C_n(x) = S_n(x) - S_{n-2}(x) \quad . \tag{11.11}$$

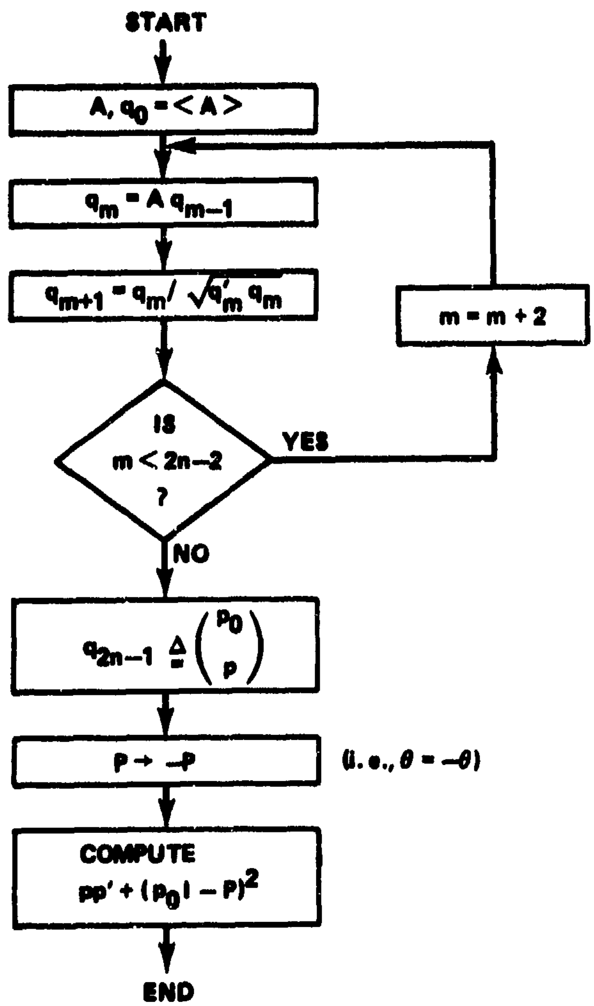

Figure 1. Single quaternion small angle recursion and conversion to an Euler matrix.

XII. ITERATION OF PRODUCT QUATERNIONS

Suppose the quaternions producing yaw, roll, and pitch motions in the similarity transformation are symbolized by A, B, and C, respectively, and that the combined transformation is the product ABC. In Section XI, $(ABC)^{1/n}$ was the small angle approximation and iteration yielded

$$\left[(ABC)^{1/n}\right]^n = ABC \quad .$$

The exponential matrix, $\exp\begin{pmatrix} 0 & 0' \\ \vec{0} & K \end{pmatrix}$, of Section IX in the context of a triple matrix product uses an extremely large integer n such that K in $[I + (K/n)]^n$ retains only first-order terms and the iteration converges to the Euler matrix. If n is not large, then a fundamental inequation

$$(ABC)^{1/n} \neq A^{1/n} B^{1/n} C^{1/n} \qquad (12.1)$$

appears behind the scenes in the quaternion aspect.

The third possibility is motivated by the desirable equality

$$\left(A^{1/n}\right)^n \left(B^{1/n}\right)^n \left(C^{1/n}\right)^n = ABC \quad . \qquad (12.2)$$

This iteration requires three parallel iterations and a final matrix product. This results in a maximal computer, but would give the safest computation for the finite arithmetic available on a dedicated digital computer.

A flow chart summary of the primary results obtained is presented in Figure 2. The starting equation is the quaternion similarity transformation. Ascending or descending vertical arrows represent major transitions whose meanings are apparent, while horizontal arrows represent rather minor transitions. END3 is a theoretical ending, whereas END4 and END5 represent computer configurations. A solidus (/) through a directed line denotes an unrecommended transition. END numbers are numerically ordered according to the occurrence of results in this report. Unrecommended exits are assigned the last numbers, i.e., 6, 7, and 8.

END4 and END5 are possible multiple exits for this study. END4 leads to existing dedicated quaternion computers. If specifications should tighten in the future, then END5 is available for alternative consideration.

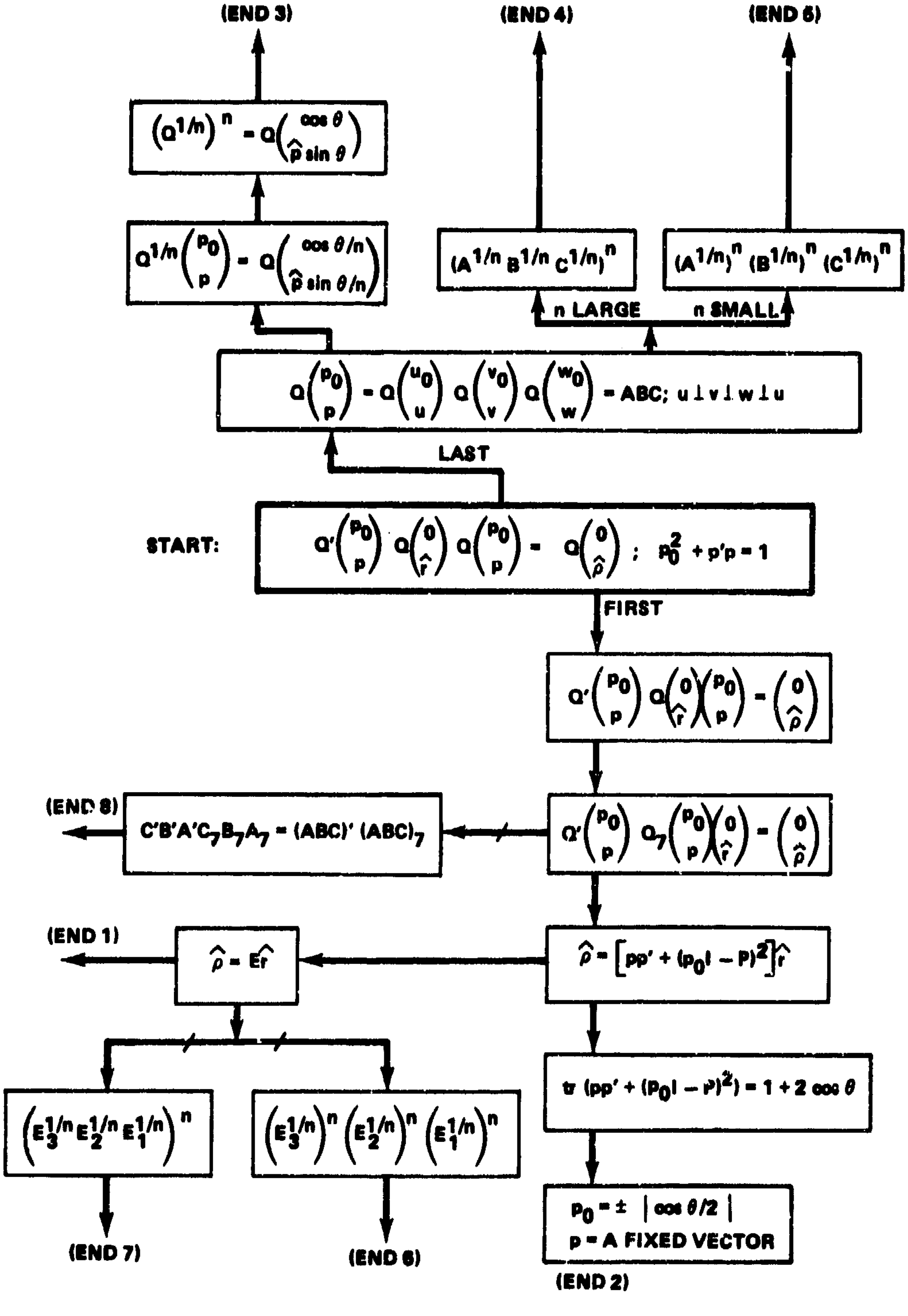

Figure 2. Flow chart for quaternion similarity transformation study.

XIII. CONCLUSIONS

Detailed quaternion applications to three-dimensional rotations of a rigid body have been presented in a bordered matrix format. There is an initial abstruseness about quaternions that eventually becomes more concrete. Recursions can be generated algebraically after introducing the nth root of a quaternion. A flow chart for the quaternion similarity transformation study summarizes procedures and results.

Two appendices contained in this report are concerned with strap down digital computers modelled by four gimbal quaternions and quaternion representation by a matrix differential equation. Numerical integration of differential equations is understood by experts; therefore, because of this and emphasis on simpler algebraic recursions, no differential equation (DE) recursions were presented. Quaternion fundamentals range over several mathematical subjects even though angular acceleration of a rigid body has not been included.

The reference list samples the extensive quaternion literature. Branets [6] has been mentioned because of the large list of European papers, books, and static and dynamic applications.

This report is oriented toward analog and digital computation and fosters fundamentals which eventually produce the following:

i) Improve computer hardware acceptance testing.

ii) Provide better formulation of computer hardware requirements.

iii) Provide minimal software emulation of hardware realizations.

Appendix A. GENERAL QUATERNION TRANSPOSES

The two transmuted quaternions can be imbedded in a set of eight general quaternion transposes. Also, there exists a natural subscript notation with simple binary number operational rules. Motivation arises from the fact that transposes of quaternions are equivalent to transformations with a diagonal matrix T.

Lemma A-1. If $T = \mathrm{diag}(1, -1, -1, -1)$ and the prime denotes matrix transpose, then

$$Q'(q) = Q(Tq)$$

and

$$T^2 = I = T^0 \quad .$$

In general, T can also be a left multiplier or a right multiplier of Q as in

$$Q_1(q) = T^0 Q(T^0 q)\ T^1 = Q(q)\ T \quad .$$

The usual binary representation of 1, $[1]_2 = 001$, immediately locates T in a position accordingly because 1 implies the presence of T and 0 implies the absence of T in a particular bit position.

It is natural to consider the subscript as an operation; namely, the generalized quaternion transpose

$$Q_i(p) = [Q(p)]_i \quad .$$

The next task is to determine k in

$$\left[Q_i(p)\right]_j = \left[Q_j(p)\right]_i = Q_k(p)$$

if i, j are given from the set of indices 0, 1, 2, 3, 4, 5, 6, 7.

Lemma A-2. If

$$[i]_2 = \alpha_2\alpha_1\alpha_0 \quad ,$$

$$[j]_2 = \beta_2\beta_1\beta_0 \quad ,$$

$$[k]_2 = \kappa_2\kappa_1\kappa_0 \quad ,$$

and $\forall$ is the term-by-term sum-modulo-2 of a pair of binary words, then

i) $\kappa_2\kappa_1\kappa_0 = (\alpha_2 \oplus \beta_2)(\alpha_1 \oplus \beta_1)(\alpha_0 \oplus \beta_0) = [i]_2 \forall [j]_2$

ii) $k = 2(2\kappa_2 + \kappa_1) + \kappa_0$

gives the decimal number k in $[Q_i(p) = Q_k(p)$.

Table A-1 displays k for given i and j. Here, it is desirable to shorten the precise $\left[[i]_2 \forall [j]_2\right]_{10}$ to $i\forall j$. A typical dogleg path for determination of $k = 4\forall j$ is shown by a pair of heavy lines. Some simple arithmetic representations of $i\forall j$ are also noted in Table A-1. It is natural to consider the fundamental subset of generalized transposed quaternion as $Q_7(p)$, $Q_4(p)$, and $Q_1(p)$.

Lemma A-3. The complete set of transposed quaternions can be determined from the fundamental subset Q_7, Q_4, and Q_1. The complete set can be conducted as follows:

i) Q_7, Q_4, Q_1 ;

ii) $\left[Q_7\right]_2, \left[Q_4\right]_2, \left[Q_1\right]_2$ yield Q_5, Q_6, and Q_3, respectively;

iii) $Q_2 = Q'$ and $Q_0 = Q$ are implicit in the construction of i) and ii).

The quaternion meanings of $Q_7(p)$ and $Q_1(p)$ have already been presented in a lead vector interchange context. The meaning of $Q_4(p)$ is that it is the row transformation counterpart of $Q_1(p)$.

TABLE A-1. DETERMINATION OF $i \nabla j = k$

	i \ j	0	1	2	3	4	5	6	7
	0	0							
	1	1	0				k		
	2	2	3	0					
	3	3	2	1	0				
START:	4	4	5	6	7	0			
	5	5	4	7	6	1	0		
	6	6	7	4	5	2	3	0	
	7	7	6	5	4	3	2	1	0

END

Identities:
$$\begin{cases} j \nabla j = 0 \\ 0 \nabla j = j \end{cases}$$

Arithmetic representations:
$$\begin{cases} 7 \nabla j = 7 - j \\ 4 \nabla j = (4 + j) \bmod 8 \\ 1 \nabla j = \begin{cases} j - 1 & \text{if } j \text{ is odd} \\ j + 1 & \text{if } j \text{ is even} \end{cases} \end{cases}$$

The subscript-7 operation is employed most frequently. Its transpose-like nature is now demonstrated.

Lemma A-4.

$$\left[Q(p)\ Q(q)\right]_7 = Q_7(q)\ Q_7(p) = Q_6(q)\ Q_3(p) \quad .$$

Proof:

$$\left[Q(p)\ Q(q)\right]_7 = T[Q(p)\ Q(q)]'T$$

$$= [T\ Q(Tq)]\ [Q(Tp)T], \quad (T^2 = I)$$

$$= Q_6(q)\ Q_3(p)$$

$$= (T\ Q(Tq)T)\ (T\ Q(Tp)T)$$

$$= Q_7(q)\ Q_7(p)$$

Only subscripts 2, 3, 6, and 7 have middle ONEs in their binary representations and these subscript operations "transpose" matrices as in Lemma A-4. Further investigation can proceed.

Lemma A-5.

$$[Q(p)\ Q(q)]_2 = Q_2(q)\ Q_2(p)$$

$$[Q(p)\ Q(q)]_3 = Q_2(q)\ Q_3(p)$$

$$[Q(p)\ Q(q)]_6 = Q_4(q)\ Q_2(p) \quad .$$

As before, these factorizations are not unique.

Appendix B. FOUR GIMBAL QUATERNIONS

Quaternion inverse equation variants lead to two error quaternions applicable to the remote control of two rotating rigid bodies. A further consequence is the four gimbal quaternion, a mathematical model for a strap down digital computer that replaces expensive mechanical gimbals.

The motivating quaternion equation is

$$Q'\begin{pmatrix}\cos\theta\\ \hat{p}\sin\theta\end{pmatrix} Q\begin{pmatrix}\cos\theta\\ \hat{p}\sin\theta\end{pmatrix} = Q\begin{pmatrix}1\\ \vec{0}\end{pmatrix} \quad . \tag{B-1}$$

Replace the first lead vector with another lead vector and form the quaternion difference

$$Q\begin{pmatrix}1\\ \vec{0}\end{pmatrix} - Q'\begin{pmatrix}\cos\psi\\ \hat{q}\sin\psi\end{pmatrix} Q\begin{pmatrix}\cos\theta\\ \hat{p}\sin\theta\end{pmatrix} \quad ,$$

where $(\cos\theta, \hat{p}\sin\theta)'$ is the reference lead vector.

Lemma B-1.

$$Q\begin{pmatrix}1\\ \vec{0}\end{pmatrix} - Q'\begin{pmatrix}\cos\psi\\ \hat{q}\sin\psi\end{pmatrix} Q\begin{pmatrix}\cos\theta\\ \hat{p}\sin\theta\end{pmatrix}$$

$$= Q\begin{pmatrix}1 - \cos\psi\cos\theta - (\hat{q}\cdot\hat{p})\sin\psi\sin\theta\\ \hat{q}\sin\psi\cos\theta - \hat{p}\cos\psi\sin\theta + (\hat{p}\times\hat{q})\sin\psi\sin\theta\end{pmatrix}$$

$$= Q'\begin{pmatrix}\cos\theta - \cos\psi\\ \hat{p}\sin\theta - \hat{q}\sin\psi\end{pmatrix} Q\begin{pmatrix}\cos\theta\\ \hat{p}\sin\theta\end{pmatrix} \quad . \tag{B-2}$$

Understanding can be furthered by making a side calculation for the special case $\hat{q} = \hat{p}$. Differences of angle will appear in such a way that all lead vector components are zeroed when the two rigid bodies are aligned and have identical angles. This result reinforces Ickes' [4] practical goal of applying quaternions to aligning two rotating systems.

Replacing one lead vector with an orthogonal lead vector in the left-hand side of Equation (B-1) leads to a complementary identity,

$$Q'\begin{pmatrix} \cos\theta \\ \hat{p}\sin\theta \end{pmatrix} Q\begin{pmatrix} -\sin\theta \\ \hat{p}\cos\theta \end{pmatrix} = Q\begin{pmatrix} 0 \\ \hat{p} \end{pmatrix} \quad . \tag{B-3}$$

Transposing both sides yields the following result.

Lemma B-2.

$$Q\begin{pmatrix} \sin\theta \\ \hat{p}\cos\theta \end{pmatrix} Q\begin{pmatrix} \cos\theta \\ \hat{p}\sin\theta \end{pmatrix} = Q\begin{pmatrix} 0 \\ \hat{p} \end{pmatrix} \tag{B-4}$$

and the quaternion product commutes. The second error quaternion complements the quaternion previously derived.

Lemma B-3.

$$Q\begin{pmatrix} 0 \\ \hat{p} \end{pmatrix} - Q\begin{pmatrix} \sin\psi \\ \hat{q}\cos\psi \end{pmatrix} Q\begin{pmatrix} \cos\theta \\ \hat{p}\sin\psi \end{pmatrix}$$

$$= Q\begin{pmatrix} -\sin\psi\cos\theta + (\hat{q}\cdot\hat{p})\cos\psi\sin\theta \\ -\hat{q}\cos\psi\cos\theta + \hat{p}(1-\sin\psi\sin\theta) + (\hat{p}\times\hat{q})\cos\psi\sin\theta \end{pmatrix}$$

$$= Q\begin{pmatrix} \sin\theta - \sin\psi \\ \hat{p}\cos\theta - \hat{q}\cos\psi \end{pmatrix} Q\begin{pmatrix} \cos\theta \\ \hat{p}\sin\psi \end{pmatrix} \quad , \tag{B-5}$$

where $(\cos\theta, \hat{p}\sin\theta)'$ is the reference lead vector. This again leads to an all zero lead vector when the two rigid bodies are aligned.

An additional application of Lemma B-2 is the nulling of the earth's spin angle, $\psi = \omega t$, along the spin axis, $\hat{r}$, for a long range missile's geonavigation system. If $Q\begin{pmatrix} 0 \\ \dot{\hat{r}} \end{pmatrix}$, in the original similarity transformation, is replaced by a counterpart of Equation (B-3), one obtains the asimilarity transformation

$$\left[Q'\begin{pmatrix} \cos\frac{\theta}{2} \\ p\sin\frac{\theta}{2} \end{pmatrix} Q\begin{pmatrix} \sin\psi \\ \hat{r}\cos\psi \end{pmatrix}\right] Q\begin{pmatrix} \cos\psi \\ \hat{r}\sin\psi \end{pmatrix} \left[Q\begin{pmatrix} \cos\frac{\theta}{2} \\ \hat{p}\sin\frac{\theta}{2} \end{pmatrix}\right] = Q\begin{pmatrix} 0 \\ \hat{p} \end{pmatrix} \quad . \tag{B-6}$$

Lead vector selection and lead vector interchange on the asimilarity transformation yield the desired result.

Lemma B-4. The asimilarity transformation yields the four-gimbal quaternion model

$$Q'\begin{pmatrix} \cos\frac{\theta}{2} \\ p \sin\frac{\theta}{2} \end{pmatrix} Q\begin{pmatrix} \sin\psi \\ \hat{r}\cos\psi \end{pmatrix} Q_7\begin{pmatrix} \cos\frac{\theta}{2} \\ \hat{p}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\psi \\ \hat{r}\sin\psi \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\rho} \end{pmatrix} \qquad \text{(B-7)}$$

which nulls the spin angle, ψ, along the spin axis, $\hat{r}$. Three of the four gimbals are resident in the half-angle quaternions, whereas the fourth gimbal is represented by $Q\begin{pmatrix} \sin\psi \\ \hat{r}\cos\psi \end{pmatrix}$. The extraordinary event here is that the angle ψ appears on the left-hand side but disappears on the right-hand side — a hidden variable situation. From this point on, $\hat{\rho}$ is the same as given in the text.

Another scenario for four-gimbal quaternions is the spin stabilized missile. Missile spin produces desirable aerodynamic stability but undesirably spins radar homing data. Four-gimbal quaternions "de-spin" the data.

The simple flowchart for error and four gimbal quaternions is shown in Figure B-1.

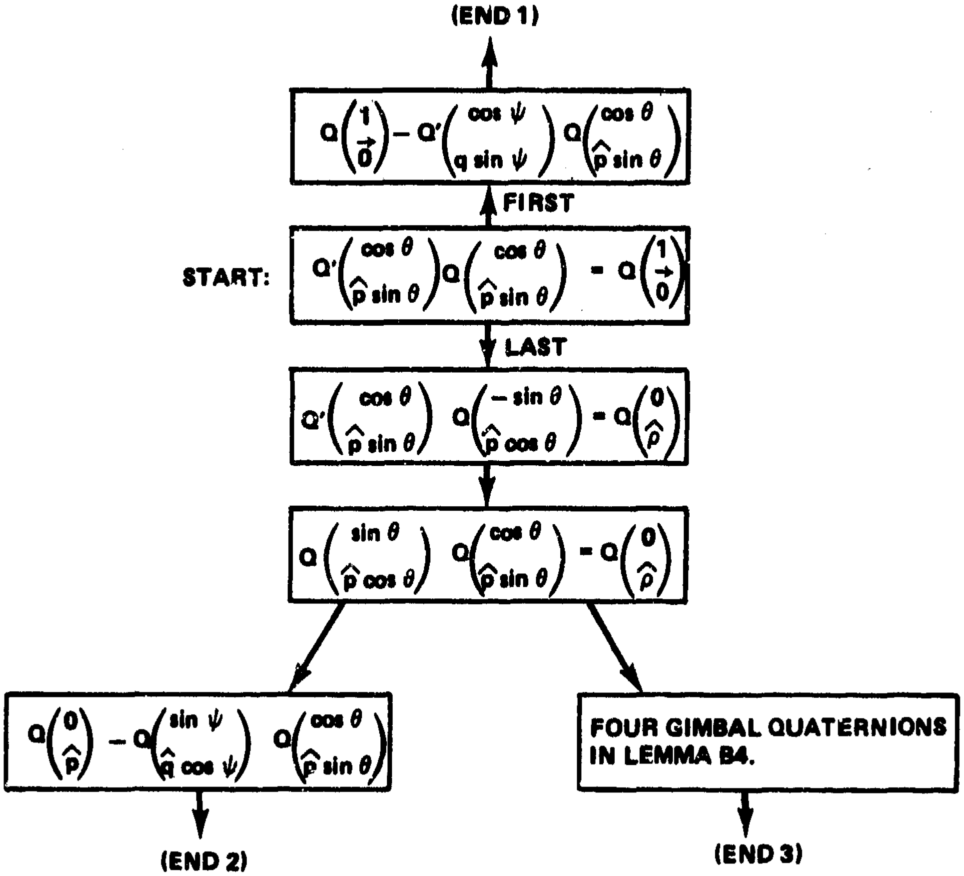

Figure B-1. Flow chart for investigation of two error quaternions and the four gimbal quaternions.

Appendix C. QUATERNIONS AND DIFFERENTIAL EQUATIONS

The matrix DE aspect of four gimbal quaternions is presented. A simple computation is considered first.

Lemma C-1. If

$$\dot{X}(t) = X(t)\, Q\begin{pmatrix} 0 \\ r \end{pmatrix}$$

and

$$X(0) = Q_7\begin{pmatrix} \cos \frac{\theta}{2} \\ \hat{p} \sin \frac{\theta}{2} \end{pmatrix} \quad ,$$

where r is a constant 3-vector, $\omega = \|r\|$, X(t) is a 4×4 matrix function of time, $0 \leq t < \infty$, then

i) $$X(t) = Q_7\begin{pmatrix} \cos \frac{\theta}{2} \\ \hat{p} \sin \frac{\theta}{2} \end{pmatrix} Q\begin{pmatrix} \cos \omega t \\ \hat{r} \sin \omega t \end{pmatrix}$$

ii) $$\langle X(t) \rangle = Q_7\begin{pmatrix} \cos \frac{\theta}{2} \\ \hat{p} \sin \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos \omega t \\ \hat{r} \sin \omega t \end{pmatrix}$$

Proof:

1) $$\omega^2 \triangleq r_0^{\,2} + r'r = r'r \quad ,$$

2) $$Q^2\begin{pmatrix} 0 \\ r \end{pmatrix} = -\omega^2 I \quad ,$$

3) $$\ddot{X} + \omega^2 X = 0 \quad ,$$

4) $$X(t) = X(0)\left[I \cos \omega t + Q\begin{pmatrix} 0 \\ r \end{pmatrix} \frac{\sin \omega t}{\omega} \right] \quad ,$$

5) $$X(t) = Q_7\begin{pmatrix} \cos \frac{\theta}{2} \\ \hat{p} \sin \frac{\theta}{2} \end{pmatrix} Q\begin{pmatrix} \cos \omega t \\ \hat{r} \sin \omega t \end{pmatrix} \quad ,$$

6) $\langle X(t)\rangle = Q_7\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\omega t\\ \hat{r}\sin\omega t\end{pmatrix}$

7) End of proof.

It is apparent that the matrix and vector portion of the four-gimbal equation with $\psi = \omega t$ have been represented by matrix differential equations. It suggests that the two leftmost matrices be computed next as the slightly more difficult case.

<u>Lemma C-2.</u> If

$$\dot{Y}(t) = Y(t)\, Q\begin{pmatrix}0\\ r\end{pmatrix}$$

and

$$Y(0) = Q'\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix} \quad ,$$

where r is a constant 3-vector, $\omega = \|r\|$, $0 \le t < \infty$, then,

i) $Y(t) = Q'\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix} Q\begin{pmatrix}\cos\omega t\\ \hat{r}\sin\omega t\end{pmatrix}$

ii) $-\omega\int_0^t Y(\tau)\, d\tau + Q'\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix} Q\begin{pmatrix}0\\ \hat{r}\end{pmatrix}$

$$= Q'\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix} Q\begin{pmatrix}-\sin\omega t\\ \hat{r}\cos\omega t\end{pmatrix} \quad .$$

Executing step ii) of Lemma C-2 and step ii) of Lemma C-1 as parallel computations followed by matrix-vector multiplication yields

$$\left[Q'\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix} Q\begin{pmatrix}-\sin\omega t\\ \hat{r}\cos\omega t\end{pmatrix}\right]\left[Q_7\begin{pmatrix}\cos\frac{\theta}{2}\\ \hat{p}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\omega t\\ \hat{r}\sin\omega t\end{pmatrix}\right] = \begin{bmatrix}0\\ \hat{p}\end{bmatrix} \quad .$$

the end result of a matrix differential equation representation of a four-gimbal platform.

It is now possible to branch into numerical integration according to Barker [7] or into analog simulation according to Mitchell [8]. Critical comments are in order for Barker's [7] numerical methodology. Barker compares the self-starting Adams-Bashforth-2 method, which belongs to numerical transform methods peculiar to electrical engineering, with

the local linearization (LL) method of numerical analysis; this is a comparison of the worst method in numerical transforms with the best method in numerical analysis. The "correct" Boxer-Thaler Z-form of numerical transforms is a much better performing numerical transform specifically applicable to $\ddot{X} + \omega^2 X = C$, $X(0) = I$, $\dot{X}(0) = Q\begin{pmatrix} 0 \\ r \end{pmatrix}$, which is identical to Barker's problem. A best-with-best comparison results in a fairer contest and one can conjecture that the LL algorithm will not be overwhelmingly superior.

Analog computer integration is rather easy. Past analog and digital simulations require representation of the quaternion with orthogonal lead vector in Lemma C-2.

A slight disadvantage of the DE representation of quaternions in that ω is dependent on $\|r\|$, whereas in the algebraic context ω and $\|r\|$ are independent.

REFERENCES

1. Whittaker, E. T., Analytical Dynamics, Fourth Edition, Dover New York, pp. 2-8.

2. Robinson, A. C. On the Use of Quaternions in Simulation of Rigid Body Motion, Project No. 7060, Wright-Patterson Air Force Base, Ohio, March 1960, pp. 1-89, WADC Technical Report 58-17.

3. Bean, W. C., Operational Use of Quaternions in the Description of Rigid Body Motion, NASA Manned Spacecraft Center, Houston, Texas, October 1969, pp. 1-18, MSC Internal Note No. 69-FM-237.

4. Ickes, B. P., "A New Method for Performing Digital Control System Attitude Computations Using Quaternions," AIAA Guidance, Control, and Flight Dynamics Conference, Pasadena, California, August 1968, (AIAA Paper No. 68-825), pp. 1-8.

5. Lanczos, C. Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$, US National Bureau of Standards, US Government Printing Office, Washington, D. C., pp. 1-161, 1952.

6. Branets, V. N., and Shmyglevskiy, I. P., Application of Quaternions to Rigid Body Rotation Problems, Scientific Translation Service, (Translated from the Russian in 1974), National Aeronautics and Space Adminstration, Washington, D. C. p. 352., 1973. Report No. NASA-TT-F-15414.

7. Barker, L. E., Jr., Bowles, R. L., and Williams, L. H., Develcoment and Application of a Local Linearization Algorithm for the Integration of Quaternion Rate Equations in Real-Time Flight Simulation Problems, NASA Langley Research Center, Langley, Virginia, NASA Technical Note D-7347.

8. Mitchell, E. E. L., and Rogers, A. E., "Quaternior Parameters in the Simulation of a Spinning Rigid Body," Simulation, Vol 4., No. 6, June, 1965, pp. 390-396.

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